Simplified equations for transient center temperature prediction in solids during short time heating or cooling

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Ramaswamy, H.S. and Sreekanth, S. 1999. **Simplified equations for transient center temperature prediction in solids during short time heating or cooling**. Can. Agric. Eng. **41**:059-064. The transient temperature of solids subjected to convective heating at the surface can be approximated, with reasonable accuracy, by the first term of an infinite series for long-time heating/cooling processes ($Fo \geq 0.2$). For short-time heating/cooling ($Fo < 0.2$), errors involved in the above approximation are quite high, increasing from about 1% at $Fo = 0.2$ to 14-31% at $Fo = 0.05$ to a maximum of 27-100% at $Fo = 0$, depending on the particle shape. Simplified equations, using first term approximation coefficients, were developed for estimating the initial curved portion of the semi-logarithmic heating/cooling curves. Initially, a regression model for the correction factor (φ) was developed for $Bi = \infty$ as a function of Fourier number (Fo). Subsequently, an additional correction factor (Ψ) was developed as a function of Biot number to accommodate finite Biot numbers. The errors in predicted temperatures using the developed equations were less than 2.5% as compared with the infinite summation series.

La température transitoire de nourritures solides soumises au chauffage par convection à la surface peut être approximée, avec exactitude raisonnable, en utilisant le premier terme d'une serie infinie pour des procédés de chauffage/refroidissement de longue durée ($Fo \geq 0,2$). Pour les procédés de chauffage/refroidissement de courte durée ($Fo<0,2$), les erreurs impliquées dans l'approximation précitée sont tout à fait élevées croissant d'environ 1% à $Fo = 0,2$ à 14-31% à $Fo = 0,05$ jusqu'à un maximum de 27-100% à $Fo = 0$, dépendantes de la forme de la particule. Des équations simplifiées se basant sur l'approximation des coefficients de premier terme ont été développées pour estimer la courbée initiale des courbes de chauffage /refroidissement semi-logarithmiques. Initialement, un modèle de régression pour le facteur de correction (φ) a été développé pour $Bi = \infty$, en tant que fonction du nombre de Fourier (Fo). Par la suite, un facteur de correction additionnel (Ψ) a été développé comme fonction du nombre de Biot pour satisfaire des nombres de Biot finis. Les erreurs des températures prédites utilisant les équations développées étaient moins que 2,5% comparé à la série de l'addition infinie.

INTRODUCTION

Estimation of the temperature history at a critical point during the heating or cooling process plays an important role in the establishment and optimization of process schedules. Theoretical formulas based on conduction heating or cooling have been applied by a number of researchers to predict time-temperature relationships in regular shaped solids under various boundary conditions (Ball 1923; Clary et al. 1971; Hayakawa 1969; Hayakawa and Ball 1971; Lenz and Lund 1978; Pflug et al. 1965; Smith et al. 1967). The basic theories dealing with transient conduction of heat in solids subjected to surface convection and radiation heat transfer have been discussed in detail by Carslaw and Jaeger (1959). The equations relating the transient temperature ratio (U) and time (t) in terms of Fourier ($Fo = \alpha t/a^2$, α is the thermal diffusivity and a is the characteristic dimension) and Biot ($Bi = ha/k$, k is the thermal conductivity of the material) numbers, with uniform initial temperature (Ti) and finite convective heat transfer coefficient (h), when immersed into a medium of constant temperature (T) is given for an infinite plate as:

$$U_P = \sum_{n=1}^{\infty} \frac{2\sin(\beta_n)}{\beta_n + \sin(\beta_n)\cos(\beta_n)} \cos(\beta_n x / l)\exp(-\beta_n^2 Fo) \quad (1)$$

where β_n is the nth positive root of $\beta \tan(\beta) = Bi$.

For an infinite cylinder:

$$U_C = \sum_{n=1}^{\infty} \frac{J_0(\psi_n r / a)}{(Bi^2 + \psi_n^2) J_0(\psi_n)} \exp(-\psi_n^2 Fo) \quad (2)$$

where ψ_n is the nth positive root of $\psi J_1(\psi) = Bi\, J_0(\psi)$.

For or a sphere:

$$U_S = \sum_{n+1}^{\infty} \frac{[\delta_n^2 + (Bi-1)^2]\sin(\delta_n)}{\delta_n^2[\delta_n^2 + Bi(Bi-1)]} \sin(\delta_n r / a)\exp(-\delta_n^2 Fo) \quad (3)$$

where δ_n is the nth positive root of $\delta \cot(\delta) = (1 - Bi)$.

For sufficiently long heating times ($Fo > 0.2$), the terms in the infinite summation series in Eqs. 1-3 converge rapidly and in most cases may be accurately approximated by the first term of the summation series (Heisler 1947). Further, at the center of a plate, cylinder, or sphere $x = r = 0$. Imposing these assumptions, Eqs. 1-3 could be simplified for center temperatures in an infinite plate (Eq. 4), infinite cylinder (Eq. 5), and sphere (Eq. 6), respectively (Heisler 1947) as:

$$U_{0P} = R_P \exp(-S_P Fo) \quad (4)$$

$$U_{0C} = R_C \exp(-S_C Fo) \quad (5)$$

$$U_{0S} = R_S \exp(-S_S Fo) \quad (6)$$

where R_P, R_C, R_S, S_P, S_C, S_S are the characteristic functions of Biot number given by:

$$R_P = \frac{2\sin(\beta_1)}{\beta_1 + \sin(\beta_1)\cos(\beta_1)}; \qquad S_P = \beta_1^2 \quad (7)$$

$$R_C = \frac{2Bi}{(Bi^2 + \psi_1^2)J_0(\psi_1)}; \qquad S_C = \psi_1^2 \quad (8)$$

$$R_S = \frac{2Bi\left[\delta_1^2 + (Bi^2 - 1)^2\right]\sin(\delta_1)}{\left[\delta_1^2 + Bi(Bi-1)\right]\delta_1}; \qquad S_S = \delta_1^2 \quad (9)$$

Equations 4-6 describe the straight line relationships on a semi-log scale between the temperature ratio and Fourier number for the three shapes with negative slopes of S_P, S_C, and S_S and intercept coefficients of R_P, R_C, and R_S. An initial lag period for the center temperature change is characteristic of any heating or cooling curve during which there will be a deviation from the above behavior. Hence, when Eqs.4-6 are employed to predict the temperature ratio during the lag period (Fo < 0.2), errors become obvious. Since, at the start of the process $U_0 = 1.0$, the maximum error for a given situation can be found from the respective intercept coefficients (R values). The maximum R value (i.e. R_∞ at Bi = ∞) for an infinite plate, infinite cylinder, and a sphere are 1.273, 1.602, and 2.0, respectively; thus, the maximum errors are as high as 27.3 - 100% at Fo=0. Calculated errors at different Fourier numbers (Table I) show that the magnitude of the prediction error (at Bi = ∞) decreases considerably as Fourier numbers increase and validates employing the first term approximation of the infinite series at Fo > 0.2 with less than 1.8% error introduced. At Fo < 0.2, however, the series (Eqs. 1-3) will have to be expanded to several terms to minimize the errors. Hayakawa (1969) presented some charts and tables for obtaining the central temperatures of canned foods during heating or cooling. However, these are limited to the geometry of a finite cylinder.

The objective of this study was to develop simplified relationships to obtain correction factors to the first term approximation in order to accurately predict the center temperatures of regular solids during the initial heating period.

METHODOLOGY

Solid center temperatures were computed using the first several terms (minimum 10) of Eqs. 1-3 and used as exact solutions (U_T) to compare the accuracy of the developed simplified equations. Center temperatures calculated using only the first term of the series (Eqs. 4-6) were represented by U_I. The correction factor (CF) was defined as (U_I-U_T) such that $U_T = U_I$ - CF. To develop the simplified equations, the conditions of infinite Biot numbers were considered first so that the corrections would depend only on the Fourier number.

Table I. Errors in center temperature ratio due to first term approximation at Fo < 0.2 and Bi = ∞.

Fourier No.	Infinite plate	Infinite cylinder	Sphere
0.00	27.3	60.2	100.0
0.05	14.2	24.5	31.1
0.10	6.8	9.2	9.1
0.15	3.5	3.7	3.0
0.20	1.8	1.6	1.0

Subsequent to this, the correction factors were modified to accommodate other Biot numbers.

Generalized relationship at Bi = ∞

An infinite Biot number, at which the errors due to first term approximation were maximum, was considered initially such that the correction factors would depend only on the Fourier number. The theoretical and first term approximation center temperature ratios, U_T and U_I, were calculated for each of the three geometries at Fourier number intervals of 0.0002 up to Fo = 0.02, 0.002 up to Fo = 0.20, and at 0.02 intervals, thereafter up to Fo = 1.75 (a total of 250 observations). A normalized correction factor function ϕ, was defined as:

$$\phi = R_\infty - \frac{U_I}{U_T} \quad (10)$$

such that the minimum value of ϕ was zero (at Fo=0), and the maximum value was (R_∞-1) because at Fo > 0.2, U_I was essentially equal to U_T with R_∞ representing the R value at infinite Biot number. The function ϕ, thus defined, was similar to an arc tangent or a hyperbolic function of Fo. A stepwise forward multiple regression technique was employed to evaluate ϕ as a function of Fo in the general form:

$$\phi = \frac{a_1}{Fo^{1/k}} + a_2 \arctan(lFo) + a_3 \arctan(mFo) + a_4 \arctan(nFo) + a_5 \quad (11)$$

where a_1, a_2, a_3, a_4, and a_5 were the regression coefficients to be evaluated with k, l, m, and n as parameters. Here, k was an integer from 1 to 10 and l, m, and n were integers from 1 to 10 and multiples of 10 thereafter up to 200. Although a number of factors were significant at $p < 0.05$, only the top four were included for simplicity. The predicted temperature ratio (U_p) was then calculated by rearranging Eq. 10 as:

$$U_P = \frac{U_I}{R_\infty - \phi} \quad (12)$$

Relationships at finite Biot number

Since Eq. 12 was based on correction factors at Bi = ∞, it was not expected to yield accurate results for finite Biot numbers. To account for variations in the Biot numbers, a factor, Ψ, was defined such that when multiplied by the correction factor (CF)

Table II. Regression parameters for the correction factor function, φ, for regular solids.

Parameter	Infinite plate	Infinite cylinder	Sphere
a_1	-0.02290	-0.1411	-0.1639
a_2	-0.07353	-0.2388	-0.3897
a_3	0.8629	1.6267	3.5503
a_4	-0.7113	-1.1710	-2.9052
a_5	0.1608	0.3788	0.7266
k	3	6	4
l	3	5	6
m	30	30	40
n	40	40	50
r^2	0.9996	0.9999	0.9999
S.E.	0.001984	0.001728	0.003138

at infinite Biot number, it would give the correction factor (CF') for the given Biot number. With Ψ so defined, the new U_p' could be obtained as:

$$CF' = (CF)\Psi = (U_1 - U_P)\Psi \tag{13}$$

$$U_P' = U_1 - (U_1 - U_P)_{Bi=\infty}\Psi \tag{14}$$

Ideally, Ψ should be a function of Biot number converging to unity at higher values of Bi. By careful examination of the extent of errors at different Biot numbers, Ψ was found to be related to the R values and a function of the type Bi exp(-Bi/C) with a maximum at Bi = C. A stepwise forward multiple regression was performed on R values and the Bi exp(-Bi/C) factors and the Ψ function was described in general as:

$$\Psi = b_1 R + b_2 Bi \exp(-Bi / C) + b_3 \tag{15}$$

where b_1, b_2, and b_3 were the regression coefficients to be evaluated with C being the most significant integer from 1 to 10.

Numerous examples can be found in the literature dealing with heat transfer in foods that demonstrate excellent correspondence between the finite difference simulations and experimental heating or cooling curves (Teixeira et al. 1969, 1975; Hayakawa 1969; Ramaswamy and Ghazala 1990; Ghazala et al. 1991, 1995). In this study, therefore, the analytical solutions were only compared against numerical simulations due to 1) the previously established correlations between the numerical and experimental time-temperature data and 2) the ease of representing the generalized data on dimensionless form for demonstrating the validity at the same time.

RESULTS and DISCUSSION

The linear regression parameters for the correction factor, ϕ, are given in Table II for the infinite plate, infinite cylinder, and sphere. The coefficients of determination (R^2) were greater than 0.999 for all three cases. The mean errors and their standard deviations (in parenthesis), while using the developed equation for calculating ϕ, were 2.1% (2.6%) for the infinite plate, 0.6% (0.64%) for the infinite cylinder, and 0.49% (0.4%) for the sphere, respectively. The mean errors and their standard deviations in the predicted temperature ratio as compared to the infinite series were comparatively much lower: 0.35% (0.51%), 0.15% (0.1%), and 0.22% (0.16%), respectively, for the three geometries. Figure 1 illustrates the excellent agreement between the original and predicted center temperature ratios for all three geometric shapes between Fourier numbers from 0.05 to 0.20. While points in the figure represented the theoretical center temperature ratios (U_T), the curves were obtained using Eqs. 11-12 with the appropriate coefficients from Table II (U_1 was calculated using Eqs. 4-9 employing the R and S values at infinite Biot numbers: 1.273 and 2.467, respectively for the finite plate, 1.602 and 5,783 for the infinite cylinder, and 2 and 9.87, respectively, for the sphere).

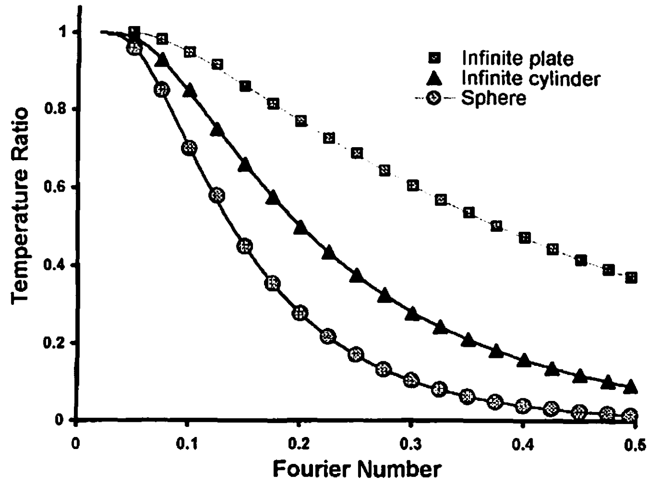

Fig. 1. Validation of the developed model for infinite Biot number situations.

Cases of negligible surface resistance

In heating or cooling processes involving Biot numbers above 40, the resistance at the product surface has been considered to be negligible compared with the internal resistances (Heldman 1975). This situation is frequently encountered in thermal processing operations where cans are heated using steam as the heating medium. It was pointed out elsewhere (Ramaswamy et al. 1982) that, although these situations were characterized by relatively large surface heat transfer coefficients, complete negligence of the surface resistance to heat transfer could introduce considerable errors (9-32% at Fo=1.0, by assuming the Biot number to be infinity when it was actually 50) and it was found that larger errors occur at higher Fourier numbers. Simplified relationships were also reported in the above studies to evaluate the R and S values for any Biot number in the range, 0.02 to 200, for an infinite plate, infinite cylinder, and a sphere.

From Table III, it can be observed that Eqs. 11-12 could be employed to predict the temperature ratio with reasonable accuracy at Biot number above 30. The prediction errors at the center of the infinite plate, infinite cylinder, and a sphere, while using Eqs. 11 and 12 at various Biot numbers, are summarized

Table III. Center temperature prediction erors1 while using φ function [Bi = ∞].

Biot number	Infinite plate Error (%)	Infinite cylinder Error (%)	Sphere Error (%)
0.1	0.3-10.1	0.1-16.3	0.0-19.5
0.5	0.1-6.9	0.1-12.0	0.0-15.1
1.0	0.1-4.1	0.1-7.9	0.1-10.5
5.0	0.8-2.0	0.8-3.1	0.7-3.1
10.0	0.4-1.7	0.6-2.8	0.6-3.9
20.0	0.2-1.3	0.5-2.0	0.5-2.8
30.0	0.1-1.0	0.4-1.5	0.5-2.2
50.0	0.0-0.8	0.2-1.1	0.2-1.5
100.0	0.0-0.6	0.0-0.7	0.0-0.9
200.0	0.0-0.5	0.0-0.5	0.0-0.6
∞	0.0-0.4	0.0-0.3	0.0-0.6

[1] Fourier number, 0.05-0.20

in Table III. The prediction errors as compared to the original expansion of the infinite series were generally less than 2% at Bi > 30. This finding, in fact, reinforces the observation made by Ramaswamy et al. (1982) that the error in obtaining the temperature histories assuming the Biot number to be infinity (when it is actually not) increases with Fourier number. Hence, the correction factor function, ϕ, can be conveniently employed to predict the central temperatures during the initial cooling or heating processes using Eq. 12 when the surface resistance to heat transfer is negligible. Furthermore, the factor (R_∞-ϕ) in Eq. 12 converges to unity at Fo > 0.2, and therefore, the first term approximation situation at Fo > 0.2 is not affected by the inclusion of (R_∞-ϕ). Thus, Eqs. 11-12 could be used for all Fourier numbers > 0.05 when the surface resistance is negligible.

Cases of finite internal and external resistances

Heldman (1975) classified the region bound by 0.1 < Bi < 40 to include situations characterized by appreciable internal as well as surface resistances. An examination of Table III reveals that the center temperatures cannot be accurately predicted using Eq. 12 when the associated Biot numbers are below 30. The errors are as high as 10-20% at a Biot number of 0.1. This was expected, because ϕ was developed employing correction factors at infinite Biot number. An additional Biot number variability factor, Ψ, was therefore found necessary to account for variations in ϕ with respect to Biot numbers.

Since Ψ was defined as a multiplication factor (Eqs. 13-14), it was essential that it converge to unity at higher Biot numbers. Table III shows the errors to increase at lower Biot numbers. A factor in the form of R/R_∞ with a maximum value of 1 at an infinite Biot number and a limiting value of $1/R_\infty$ at lower Biot numbers was initially used as a multiplication factor to compensate for the increase in errors. This resulted in a considerable reduction in the extent of errors, but also gave error maximums at intermediate Biot numbers of 2 to 6. Another function Bi exp(-Bi/C) which possessed a maximum at Bi = C was added to compensate for these intermediate higher errors. The final equations for Ψ derived from multiple regression analysis of R values and Bi exp(-Bi/C), C being an integer from 1 to 10, were as follows: for the infinite plate (Eq. 16), infinite cylinder (Eq. 17), and sphere (Eq. 18), respectively:

$$\Psi_P = 3.615R + 0.1187Bi\ \exp(-Bi/2) - 3.597 \qquad r^2 = 0.999 \tag{16}$$

$$\Psi_C = 1.552R + 0.1830Bi\ \exp(-Bi/4) - 1.468 \qquad r^2 = 0.983 \tag{17}$$

$$\Psi_S = 0.9449R + 0.2252Bi\ \exp(-Bi/4) - 0.8669 \qquad r^2 = 0.993 \tag{18}$$

The recalculated prediction errors over the broad region of Biot numbers from 0.1 to ∞, as calculated by the developed equations including φ and Ψ, are presented in Table IV. The errors for the cases of finite surface and internal resistances (0.1 < Bi < 40) were below 2.5% for all the geometries. It can also be noted from Table IV, that the center temperature prediction errors for the high Biot number region was only slightly affected by the inclusion of the Biot number variability factor Ψ. The error was less than 2.5% for all situations in the range of Biot number 0.1 to 200 and Fourier numbers 0.05 to 0.20. The errors were slightly higher at Bi = ∞ due to the discrepancy in the convergence of Ψ to exact unity.

Table IV. Center temperature prediction errors1 when employing the developed ϕ and Ψ functions.

Biot number	Infinite plate Error (%)	Infinite cylinder Error (%)	Sphere Error (%)
0.1	0.1-0.3	0.1-1.1	0.1-1.1
0.5	0.8-1.4	0.3-1.2	0.2-0.9
1.0	1.1-2.2	0.7-2.2	0.5-1.9
5.0	0.8-2.5	0.6-2.0	0.8-2.2
10.0	0.8-2.0	0.2-1.3	0.3-1.5
20.0	0.2-1.3	0.5-1.5	0.5-2.1
30.0	0.1-1.0	0.4-1.2	0.5-1.8
50.0	0.0-0.8	0.2-0.8	0.1-1.0
100.0	0.0-0.5	0.0-0.3	0.0-0.5
200.0	0.0-0.4	0.0-0.3	0.0-0.5
∞	0.0-0.4	0.0-1.3	0.0-1.9

[1] Fourier number, 0.05-0.20

This category represents test conditions under which most practical data are obtained. The performance of the analytical models for this situation is shown in Fig. 2 for each geometry for Biot numbers between 1 and 50. The points on the curve represent discrete values obtained from the analytical solutions while the continuous line represents prediction from the numerical simulation (equivalent of experimental curves). Again excellent correspondence can be seen between predicted and simulation values for all Bi between 1 and 50 with Fo above 0.05. It should be noted that the models deviate significantly from real values at Fo <0.05 (especially for Bi<10)

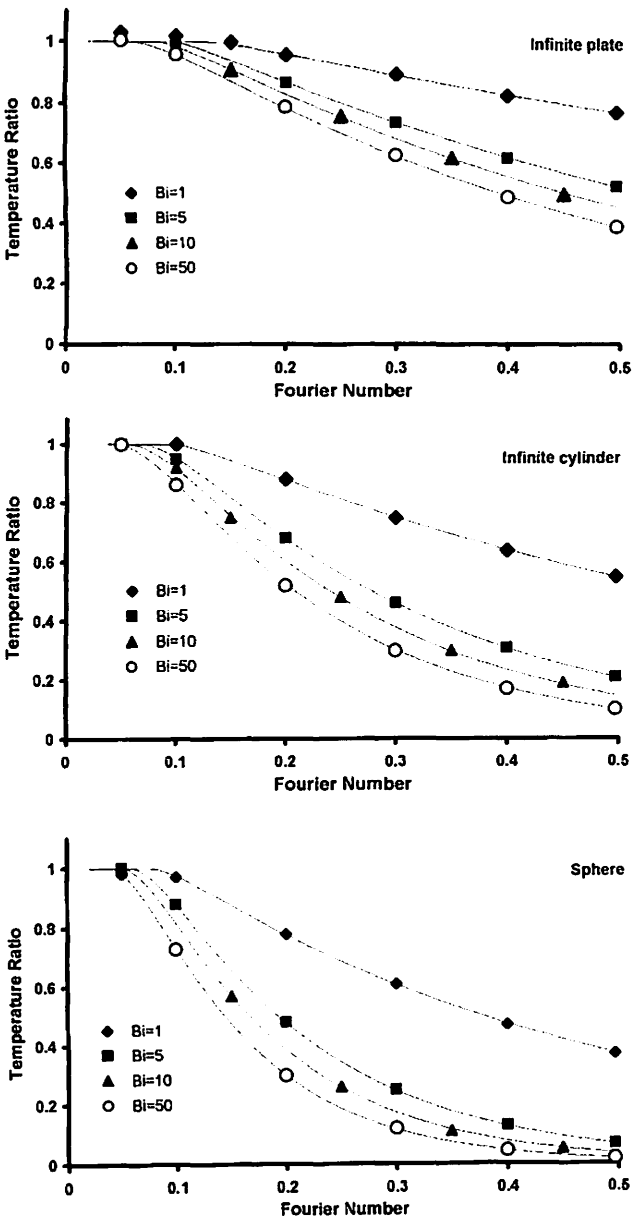

Fig. 2. Comparison between the developed simplified models versus the exact numerical solution for transient center temperatures in regular solids at various Biot and Fourier numbers.

since these were not used in regression analyses. For these situations, the center temperatures hardly show any response until an Fo of 0.05 is reached; U can thus be assumed to remain at 1.0 with less than 2% error introduced.

Cases of negligible internal resistance

These were reported to be characterized by low Biot numbers of < 0.1 (Heldman 1975). An analysis of the computed temperature ratios at the center of the three geometries indicated that up to a Fourier number of 0.2, the errors in using the first term approximation equations was less than 2%. Hence it was considered that additional correction factors were unnecessary for these situations.

CONCLUSIONS

Simplified relationships have been developed to predict the dimensionless center temperature ratio, in regular conductive solids undergoing convection heat transfer at the surface, during the initial heating or cooling period (Fo < 0.2). The errors involved in using the developed equations were less than 2.5% compared with about 31% in the same range of Fo using first term approximation of the infinite series. These equations are simple to use even with hand-held calculators and are non-iterative. The present method provides a relatively rapid and simple method to evaluate the center temperatures in a food of known thermal properties under short time heating/cooling processes.

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